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Translated by L. K.

UDC 539.3

MAXIMUM PRINCIPLE IN PROBLEMS OF OPTIMAL DESIGN OF REINFORCED SHELLS FOR NONUNIFORM LOADING

PMM Vol. 40, № 3, 1976, pp. 569-573
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(Received July 18, 1974)

We use the Pontriagin maximum principle to solve the problem of weight-optimal reinforcement of a shell acted upon by a nonuniform axisymmetric external load. When the problems of optimizing the constructional parameters and restrictions are formulated, a class of solutions is always indicated and the optimal solution is chosen from this class. Earlier, the authors of [1] used the Pontriagin maximum principle to obtain the optimal distribution of material along the length of the shell under a nonuniform load. Below we solve a similar problem with a preliminary condition that the shell has constant thickness and transverse reinforcing supports.

We consider a semimembrane model of the shell, in which the axis of the frame is assumed to coincide with the median surface, and be inextensible. After separating the variables, the equation of stability yields

$$\frac{d^4\varphi_n}{dx^4} - \alpha_n^{\ 4}\varphi_n = 0, \qquad \alpha_n^{\ 4} = \frac{q(x)R}{E\delta} n^4(n^2-1) - \frac{D}{E\delta R^2} n^4(n^2-1)^2 \tag{1}$$

The conditions of compatibility of deformations must hold at the points of the frame supports $x = l_1, l_2, \ldots, l_m$. Taking into account the fact that a passage across the frame is accompanied by a jump in the shearing and the longitudinal forces, we obtain the relations connecting the stresses and displacements in the form

$$\varphi_+ = \varphi_-, \qquad \varphi_+' = \varphi_-', \qquad \varphi_+'' = \varphi_-'' + \gamma_2\varphi', \qquad \varphi_+''' = \varphi_-''' - \gamma_1\varphi \tag{2}$$

$$\gamma_1 = \frac{n^4(n^2-1)}{E\delta R}\left[\frac{E^\circ I_x(n^2-1)}{R^2} - N^\circ\right], \qquad \gamma_2 = \frac{n^2(n^2-1)}{E\delta R^3}\left(\frac{n^2}{E^\circ I_z} + \frac{1}{GI_*}\right)^{-1}$$

Here and henceforth we adopt the following notation: l, R and δ are the length, radius and thickness of the shell, respectively; E and E° denote the moduli of elasticity of the shell material and the frame; I_x, I_z and I_* are the moments of inertia of the frame; D is bending strength and N° denotes the stress within the frame. The expressions for γ_1 and γ_2 were taken from [2].

Let us specify that the frame is rectangular ($b \times h$) and write the relations (1) and (2) in phase coordinates [3]

$$y_1^\cdot = y_2, \quad y_2^\cdot = y_3 + \frac{c_1}{\delta}\left(\frac{c_2}{bh} + \frac{c_3}{bh^3\gamma}\right)^{-1} y_2 u, \quad y_3^\cdot = y_4 - \frac{a_1}{\delta}(a_2 bh^3 - a_3)\, y_1 u \tag{3}$$

$$y_4^\cdot = \alpha_n^4 y_1 \quad (y_1 = \varphi_n,\ y_2 = \varphi_n^\cdot,\quad y_3 = \varphi_n^{\cdot\cdot},\ y_4 = \varphi_n^{\cdot\cdot\cdot})$$

Here c_i and a_i are constants determined from (1) and (2), and u is a function describing the engaging of the frame in the work

$$u = \begin{cases} 1, & x = l_k \\ 0, & x \neq l_k \end{cases}, \quad k = 1, 2, \ldots, m$$

We formulate the boundary conditions in accordance with the conditions of support. For a hinged support we have

$$y_1(0) = y_1(l) = y_3(0) = y_3(l) = 0 \tag{4}$$

We require to find the values of b, h, δ and l_k for which the condition

$$I = \int_0^l (2\pi R\delta + bhu)\, dx = \min \tag{5}$$

holds.

Let us use the formulation for the theory of optimal processes to study a process with movable ends and parameters, and fixed time. For such processes the usual maximum principle holds

$$\frac{\partial y_i}{\partial x} = \frac{\partial H}{\partial \psi_i}, \quad \frac{\partial \psi_i}{\partial x} = -\frac{\partial H}{\partial y_i}, \quad i = 1, 2, \ldots, p \tag{6}$$

$$H(\mathbf{y}, \boldsymbol{\psi}, \mathbf{u}, x) = \sum_{i=1}^{p} f_i \psi_i, \quad H(\mathbf{y}, \boldsymbol{\psi}, \mathbf{u}, x) = M(\mathbf{y}, \boldsymbol{\psi}, x)$$

$$(M = \sup_{u \in D} H(\mathbf{y}, \boldsymbol{\psi}, \mathbf{u}, x)), \quad \psi_0 \leqslant 0, \quad \psi_0 = \text{const}$$

and the following relation is valid:

$$\sum_{i=0}^{p} \psi_i(l) \int_0^l \frac{\partial f_i(y, u, \omega)}{\partial \omega_\rho}\, dx = 0, \quad \rho = 1, 2, 3, \quad \omega = \{b, h, \delta\} \tag{7}$$

We write the function $H(y, \psi, u, x)$ for the problem under consideration

$$H = A + \left[\psi_0 bh + \psi_2 \frac{c_1}{\delta}\left(\frac{c_2}{b^3 h} + \frac{c_3}{bh^3\gamma}\right)^{-1} y_2 - \psi_3 \frac{a_1}{\delta}(a_2 bh^3 - a_3)\, y_1\right] u$$

From the condition that $H(y, \psi, u, x) = M(y, \psi, x)$ we obtain

$$\psi_0 bh + \psi_2 y_2 \frac{c_1}{\delta}\left(\frac{c_2}{b^3 h} + \frac{c_3}{bh^3\gamma}\right)^{-1} - \psi_3 y_1 \frac{a_1}{\delta}(a_2 bh^3 - a_3) = 0 \tag{8}$$

since $u = \begin{cases}1\\0\end{cases}$ and $\psi_0 \leqslant 0$, $\psi_0 = \text{const}$.

Further, the additional conditions (7) give the relations

$$N_1 = \psi_0 2\pi R l - \psi_3(l) \frac{a_1}{\delta^2}(a_2 bh^3 - a_3) \sum_{k=1}^{m} y_1(l_k) = 0 \tag{9}$$

$$N_2 = \psi_0 mh - \psi_3(l) \sum_{k=1}^{m} y_1(l_k) \frac{a_1}{\delta} a_2 h^3 = 0 \tag{10}$$

$$N_3 = \psi_0 bm - 3\psi_3(l) \frac{a_1}{\delta} a_2 bh^2 \sum_{k=1}^{m} y_1(l_k) = 0 \tag{11}$$

From (11) it follows that $b = 0$. Therefore, introducing the restriction $b \geqslant b_*$ we find that (10) is replaced by the inequality $N_2 \leqslant 0$ which is satisfied automatically when Eq.(11) as well as the conditions that $\psi_0 \leqslant 0$ and $b = b_*$, all hold. Eliminating $\psi_3(l) \sum_k y_1(l_k)$ from (9) and (11), we obtain the following equation for determining h:

$$(a_2 b_* h^3 - a_3)\, m - 6\pi R l \delta a_2 b_* h^2 = 0 \tag{12}$$

Substituting ψ_0 from (9) into the condition (8), we obtain the following system of equations for determining l_k:

$$3\psi_3(l) \sum_{k=1}^{m} y_1(l_k) \frac{a_2 b^2 h^3}{m} + \psi_3(l_1)\, y_1(l_1)\,(a_2 bh^3 - a_3) = 0 \tag{13}$$

$$\psi_3(l_k)\, y_1(l_k) = \text{const}, \quad k = 2, 3, \ldots, m$$

Since the conditions of the maximum principle for the type of equations considered here are necessary but not sufficient, the number m must be found by comparing several versions of the computaional method. Below we shall use the results of [4] and assume that action of a nonuniform load $q^\circ f(x)$ on the segment can be replaced by the action of a uniform load with its ordinate given by

$$q^\circ (l_k - l_{k-1})^{-1} \int_{l_{k-1}}^{l_k} f(x)\, dx$$

Then the general solution of the initial system of equations (3) for the i-th segment will have the form

$$y_1^i(x) = y_1(l_{i-1})\,\xi_1 + y_2(l_{i-1})\,\xi_2 + y_3(l_{i-1})\,\xi_3 + y_4(l_{i-1})\,\xi_4 \tag{14}$$

or

$$\mathbf{y}(x) = \prod_{k=1}^{i-1} A_k B_k A_i(x) \cdot \mathbf{y}(0) = D_i(x)\, \mathbf{y}(0), \quad \mathbf{y}(x) = \{y_1(x), y_2(x), y_3(x), y_4(x)\} \tag{15}$$

where ξ_j is the Krylov function of the first or second kind, depending on the quantity $\alpha_n{}^4$

$$A_k = \begin{Vmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \alpha_k{}^4\xi_4 & \xi_1 & \xi_2 & \xi_3 \\ \alpha_k{}^4\xi_3 & \alpha_k{}^4\xi_4 & \xi_1 & \xi_2 \\ \alpha_k{}^4\xi_2 & \alpha_k{}^4\xi_3 & \alpha_k{}^4\xi_4 & \xi_1 \end{Vmatrix}, \quad B_k = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \gamma_2 & 1 & 0 \\ -\gamma_1 & 0 & 0 & 1 \end{Vmatrix}$$

Let us set [5] $D(l) = \| a_{ij} \|$, $i, j = 1, 2, 3, 4$ and write the conditions of existence of a solution of the system supported by a hinge (14) and rigidly clamped (15)

$$a_{12}a_{34} - a_{14}a_{32} = 0$$

$$a_{13}a_{24} - a_{14}a_{23} = 0$$

To find the forms $y(x)$ we must determine $y_2(0)$ and $y_4(0)$. The constant $y_2(0)$ can be normalized. With a hinged support we have

$$y_4(0) = y_2(0)\frac{a_{12}}{a_{14}} \tag{16}$$

Since the boundary value problem (6) is selfconjugate, the vectors $y(x)$ and $\psi(x)$ are identical, therefore for a conjugate system of equations the solution becomes

$$\psi^i(x) = \prod_{k=1}^{i-1} A_k B_k A_i(x)\psi(0), \qquad \psi(x) = \{\psi_4(x), \psi_3(x), \psi_2(x), \psi_1(x)\} \tag{17}$$

Thus we have obtained the resolving system of equations (12), (13), (14) and (16) for determining the unknown parameters h, l_i and δ

$$f_k(h, l_i, \delta, q^\circ) = 0, \qquad k = 1, 2, \ldots, m+1 \tag{18}$$

The nonlinear system obtained was solved using the Newton-Kantorovich method. The initial vector $\mathbf{F} = \{\delta, l_1, \ldots, l_m, h\}$ was given, then in accordance with the scheme it was increased by the increment $\Delta\mathbf{F}$ and the matrix

$$A = \| \partial f_k / \partial F_j \|, \qquad j, k = 1, 2, \ldots, m+1$$

computed. The corrections were calculated by means of the matrix operation

$$\Delta^n \mathbf{F} = | A_{n-1} |^{-1} \mathbf{f}^{n-1}$$

where n denotes the number of the correction and f^{n-1} is the residual column of the system $f_k = 0$ at the $(n-1)$-th step.

Such an algorithm can be used in the case when an *a priori* assumption is made that the geometrical characteristics of all supports are identical. When the characteristics are different, then m functions of u_k defined in the same manner as u, are used as the control functions

$$y_1^\cdot = y_2 \qquad y_2^\cdot = y_3 + \sum_k \frac{c_1}{\delta}\left(\frac{c_2}{h_k b_k^3} + \frac{c_3}{b_k h_k^3 k_k}\right)^{-1} y_2 u_k(x) \tag{19}$$

$$y_3^\cdot = y_4 - \sum_k \frac{a_1}{\delta}(a_2 b_k h_k^3 - a_3) y_1 u_k(x), \qquad y_4^\cdot = \alpha_n^4 y_1$$

The condition $H = M$ is now transformed into an independent system of equations in l_k

$$3\psi_3(l)\, y_1(l_k)\, a_2 b_k^2 h_k^2 + \psi_3(l_k)\, y_1(l_k)(a_2 b_k h_k^3) - a_3 = 0 \tag{20}$$
$$i = 1, 2, \ldots, m$$

Eliminating ψ_0 from (9) and (11), we obtain the following dependent system of equations for h_k:

$$\sum_{k=1}^{m} (a_2 b_k h_k^3 - a_3)\, y_1(l_k) - 3a_2 b_k h_k^2 y_1(l_k) = 0, \qquad k = 1, 2, \ldots, m \tag{21}$$

Thus in addition to the condition that the determinant (14) vanishes, we obtain a sufficient number of conditions for determining all unknown parameters. The resulting system can also be solved using the Newton-Kantorovich method.

The weight of the optimal shell G was compared with the results obtained from the condition that the shell resists equally the general and the local loss of stability [6]. Table 1 (overleaf) gives the results of this comparison. Here δ° and G° denote, respectively, the thickness and the weight of the shell obtained according to the method given in [6].

Table 1

		l_i/l_1				δ/δ°	m	G°/G
$q = \text{const}$	$h = \text{const}$	1	0.666	1	—	0.7	2	1.21
	$h \neq \text{const}$	1	0.89	0.89	1	0.61	3	1.4
$q = q^\circ e^{-\alpha x}$	$h = \text{const}$	1	0.81	—	—	0.79	1	1.19
	$h \neq \text{const}$	1	0.89	1.3	—	0.7	2	1.37

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UDC 539.3: 534.231.1

GENERALIZED SOLUTIONS OF BOUNDARY VALUE PROBLEMS IN THE THEORY OF ELASTICITY FOR RANDOM LOADING

PMM Vol. 40, № 3, 1976, pp. 573-576

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(Received October 24, 1974)

In the present paper we analyze the fundamental static and dynamic boundary value problems of the theory of elasticity, for the case of random loads. We introduce and study various generalized solutions of these problems. The solutions either appear as generalized random functions (random distributions), or belong to the spaces of summable random functions analogous to the Sobolev spaces. These spaces were introduced in [1], and we make use of the imbedding theorem for the random functions proved in that paper to establish the conditions under which the classical solution exists.